J. DIFFERENTIAL GEOMETRY
40 (1994) 595-620

CONNECTION PRESERVING ACTIONS OF CONNECTED AND DISCRETE LIE GROUPS

EDWARD R. GOETZE

ABSTRACT. This paper examines connection preserving actions of a noncompact semisimple Lie group G on a compact fiber bundle and connection preserving actions of a lattice $\Gamma \subset G$ on a compact manifold. The results rely on a new technique that increases the regularity of sections of bundles naturally associated to the actions under consideration.

1. Introduction

Let M be a connected smooth n-dimensional manifold, and H a subgroup of $GL(n, \mathbb{R})$. An H-structure on M is a reduction of the full frame bundle over M to H. If we allow H to be a subgroup of $GL(n, \mathbb{R})^{(k)}$, the subgroup of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing 0, we can extend the notion of an H-structure to include reductions of higher order frame bundles to H. Given an H-structure $P \to M$, the automorphism group of P, $\mathrm{Aut}(P)$, is the subgroup of $\mathrm{Diff}(M)$ consisting of the diffeomorphisms of M whose induced action on the frame bundle preserves P. We wish to examine relationships between a Lie group G and manifolds M with H-structures such that $G \subset \mathrm{Aut}(P)$. Also, we are interested in the situation where, instead of a G action, we have only $\Gamma \subset \mathrm{Aut}(P)$, $\Gamma \subset G$ being a lattice subgroup. This case deals with the issue of the rigidity of the action of a higher rank lattice, an area of much recent research. The use of hyperbolic dynamical systems by Hurder in [7], and Katok and Lewis in [9] and [10] has produced recent results.

If we assume M is a compact manifold and G preserves a volume form on M, then the study of the ergodic theory of the action has been a successful technique in answering some of these questions. In particular, we mention Zimmer's work in [15] and [16] as examples of this technique. One drawback of this approach, however, is that the use of ergodicity provides measurable information which is often difficult to translate into meaningful information of a higher regularity. This information, which

Received August 30, 1993 and, in revised form, December 14, 1993.

typically is represented as a section of an associated bundle to a principal bundle over M, is often defined only on an open dense subset of M. Hence, an additional difficulty in applying these methods is the problem of demonstrating that such a section has completeness properties rich enough to imply useful geometric information, for example, the section is defined on all of M. See [17] for a discussion.

In §2 of this paper, we develop a technique that under suitable hypotheses allows one to improve the regularity of this information, e.g., to pass from the measurable to the smooth. The approach we take is to combine elements of the theory of hyperbolic dynamical systems and the ergodic theory of the action, in the form of Zimmer's measurable superrigidity, with geometric considerations to construct smooth sections of bundles naturally associated to M. We prove

Theorem 2.19. *Let X be a compact fiber bundle over Y with fibers F. Let G be a connected semisimple Lie group of higher rank without compact factors. Suppose G acts ergodically on X via bundle automorphisms preserving a volume density and a C^r connection such that the fibers are autoparallel. If the algebraic hull of $\alpha' : G \times X \to SL(f)$, the derivative cocycle in the fiber direction, equals $SL(f)$, then, by possibly having to pass to a finite algebraic cover of G, there exists an abelian subgroup $A \subset G$ such that the Oseledec decomposition of TF corresponding to A is parallel, C^r regular, and everywhere defined on F_x for almost every $x \in X$. In particular, there exists a C^r section of the full flag bundle over $T(F_x)$.*

Although this result assumes the algebraic hull is $SL(f)$, this assumption is much stronger than necessary, and the proof can easily be adapted to other situations. Theorem 2.20 describes a similar result for actions of lattices on manifolds.

The existence of these sections is often too weak to provide meaningful geometric results. To strengthen this information we can use either of two methods described in this paper. The first of these methods, developed in §3, is to employ $C^{(r,s)}$ Superrigidity which is a generalization of Zimmer's topological superrigidity [14]. Where applicable, use of topological or $C^{(r,s)}$ superrigidity allows us to conclude that sections of an associated bundle over M come from sections of the corresponding principal bundle over M. More specifically, if $P \to M$ is a principal bundle H bundle, and $E_V = (P \times V)/H$ is an associated bundle with Φ a section of $E_V \to M$, then topological or $C^{(r,s)}$ superrigidity ensures the existence of a section ϕ of $P \to M$ and an element $v_0 \in V$ such that $\Phi(m) = [\phi(m), v_0]$. The essential point to note is that ϕ possesses the same regularity and completeness properties as Φ. $C^{(r,s)}$ superrigidity extends topological superrigidity in that it holds in the case where one makes more delicate regularity or completeness assumptions on Φ. See the definition of $C^{(r,s)}$ regularity below.

By exploiting the algebraic properties of H, this section of the principal bundle $P \to M$ can be used to provide much stronger information about M, and we use such a section below to classify the possibilities for M under certain conditions.

The second technique, developed with Renato Feres, is discussed in §4.2. Here we analyze the local holonomy of the connection, and, if conditions are appropriate, we conclude directly from this information that our original sections have originated from sections of principal bundles. In [5], Feres uses this technique to draw similar conclusions under more general situations, illustrating the ability to generalize the techniques to broader situations.

The motivation for the development of these techniques was to analyze the following geometric problems. Suppose G is a higher rank noncompact semisimple Lie group with $\Gamma \subset G$ a lattice. Let Γ act ergodically on a compact manifold M preserving a volume density and a connection. Does this action place any restriction on the possibilities for M? More generally, let X be a fiber bundle over Y with compact fibers F. Suppose G acts ergodically on X via bundle automorphisms preserving a volume density and a smooth connection. Does the G action restrict the choices for F? If Γ is cocompact, by inducing the action of Γ to G, we find that determining the possibilities for F restricts the possibilities for M.

We use the methods described above to obtain the following results, which are presented in §4.

Theorem 4.1. *Let X, Y, F, and G be as described above, and assume the fibers in X are autoparallel. Let L be the algebraic hull for the cocycle α'.*

(i) *If L is compact, then there exists a smooth Riemannian metric g on F with respect to which F has a transitive group of isometries, i.e.,*

$$F \cong \frac{\mathrm{Isom}(F, g)}{\mathrm{Isom}(F, g)_x}.$$

(ii) *If $L = SL(f)$, where $f = \dim(F)$, then F is a torus.*

Theorem 4.4. *Let $\Gamma \subset G$ be an irreducible lattice in a higher rank semisimple Lie group without compact factors. Suppose Γ acts ergodically on a compact n-dimensional manifold M preserving a volume and a* *smooth connection. Let* $\alpha : \Gamma \times M \to SL(n)$ *be the derivative cocycle with measurable algebraic hull* L.

(i) *If* L *is compact,* Γ *acts isometrically on* M *preserving a smooth Riemannian metric.*

(ii) *If* $L = SL(n)$, *and* $\pi(\Gamma) \subset SL(n)$ *contains a lattice, where* π *is the superrigidity homomorphism, then* M *admits a torus as a finite affine cover.*

This last theorem is a generalization of the results presented in [6] where we no longer require the rather restrictive assumption that the connection has bounded parallel transport.

As a simple corollary to Theorem 4.4, we mention

Corollary 4.8. *Suppose* $SL(n, \mathbb{Z})$, $n \geq 3$, *acts ergodically on an n-dimensional manifold* M *preserving a connection and a volume density. Then* M *admits a torus as a finite affine cover.*

Since the requirement that $L = SL(f)$ is stronger than necessary in Theorem 2.19, the same holds true for these results. There should be little problem in adapting these results to hold when L is any noncompact semisimple Lie group, regardless of its type, as long as the dimension of L is comparable to the dimension of F. This point will be addressed in future work.

We wish to thank Robert Zimmer and Renato Feres for their helpful conversations and suggestions.

2. Improving the regularity of sections

Throughout this section and those that follow, we assume the reader is familiar with the elements of Zimmer's measurable superrigidity as presented in [15] and [16].

2.1. Measurable superrigidity and the multiplicative ergodic theorem. Let X be a compact fiber bundle over Y with fibers F, $\dim(X) = x$, $\dim(F) = f$. Let G be a connected noncompact semisimple Lie group of higher rank. Suppose G acts ergodically on X via bundle automorphisms preserving a connection and a volume density. A bundle automorphism is a diffeomorphism of X which factors to a diffeomorphism of Y. Naturally associated to this situation are two cocycles:

$$\beta : G \times Y \to \mathrm{Diff}(F), \qquad \alpha : G \times X \to GL(x).$$

β describes the lift of the G action on Y to X, and α is the derivative cocycle ($x = \dim(X)$). Since the G action maps fibers to fibers, α induces another natural cocycle

$$\alpha' : G \times X \to GL(f),$$

which describes how the G action maps tangent vectors in one fiber to tangent vectors in another fiber. We let L be the algebraic hull of α'.

We shall make extensive use of the following result.

Theorem 2.1 (Superrigidity for cocycles [15]). *Suppose G is a connected semisimple Lie group, $\mathbb{R}$-rank $(G) \geq 2$, with no compact factors. Let X be an irreducible ergodic G-space, H and algebraic $\mathbb{R}$-group, and $\alpha: G \times X \to H$ a cocycle with algebraic hull L. If $L_{\mathbb{R}}$ is noncompact and center free, then $\alpha \sim \pi : G \to L$.*

Remark 2.1. If $L_{\mathbb{R}}$ has a finite center Z, we apply the theorem to $L_{\mathbb{R}}/Z$, and by lifting this homomorphism we obtain a homomorphism from a finite algebraic cover of G into $L_{\mathbb{R}}$ itself (Remark 6.2 [14]). This point necessitates the inclusion of the "up to finite cover" phrases in most of our results.

Remark 2.2. We may dispense with the assumption of irreducibility if, instead, we assume every simple factor of G has higher rank.

Remark 2.3. If $\Gamma \subset G$ is a lattice acting on X as above, then the conclusion still holds, i.e., if $\alpha: \Gamma \times X \to H$ is a cocycle, there exists $\pi: G \to H$ such that $\alpha \sim \pi$. See Theorem 9.4.14 in [15].

We will use this theorem in conjunction with some classical results from ergodic theory.

Theorem 2.2. *Let M be a compact smooth manifold, $f : M \to M$ a C^1 diffeomorphism, and $\|\cdot\|$ the norm of some Riemannian metric on M. Let $\mathscr{B}$ be the σ-algebra of Borel subsets of M and let $\mathscr{M}$ be the set of all f-invariant probability measures on $\mathscr{B}$. Then there exists an f-invariant set $B \in \mathscr{B}$ such that*

(i) *$\mu(B) = 1$ for every $\mu \in \mathscr{M}$,*
(ii) *there exists a measurable f-invariant function $s : B \to \mathbb{Z}^+$,*
(iii) *there exist measurable f-invariant functions $\chi_i : B \to \mathbb{R}$, for $i = 1, 2, \cdots, s$,*
(iv) *there exists a measurable decomposition $TM|_B = E_1 \oplus E_2 \oplus \cdots \oplus E_s$ into f-invariant subbundles, and*
(v) *if $x \in B$, $V \in E_i(x) \setminus \{0\}$, $1 \leq i \leq s(x)$, then*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Tf_x^n(V)\| = \chi_i(x).$$

The objects are unique and independent of the choice of Riemannian metric.

Proof. This is Oseledec's Multiplicative Ergodic Theorem (Theorem 10.4 in [13]). q.e.d.

The decomposition is called the *Oseledec decomposition*, the functions $\{\chi_i\}$ are called the *Lyapunov exponents*, and B is called the set of *regular points*.

If A consists of a family of commuting C^1 diffeomorphisms, we can choose the Oseledec decomposition to be common to all elements of A on some conull set Λ. The Lyapunov exponents then become functions $\chi_i : A \to \mathrm{Func}(\Lambda, \mathbb{R})$.

We wish to combine superrigidity and the Multiplicative Ergodic Theorem to yield information concerning the Lyapunov exponents in the direction of the fibers in X. Let $TF \subset TX$ consist of the subbundle of vectors tangent to the fibers in X. Assuming L is not compact, superrigidity yields a measurable trivialization of $TF \cong X \times \mathbb{R}^f$ such that the G action becomes

$$g(x, V) = (gx, \pi(g)V),$$

where $\alpha' \sim \pi : G \to SL(f)$. For an abelian subgroup $A \subset G$, let $\{\chi_i(a)\}$ be the set of Lyapunov exponents corresponding to the Oseledec decomposition

$$TF_x \cong \bigoplus_j F_j,$$

the tangent space through the fiber at $x \in X$.

Proposition 2.3. *The Lyapunov exponents for $a \in A$ are $\{\log|a_i|\}$, where a_i are the eigenvalues for $\pi(a) \in SL(f)$.*

Proof. See [6]. q.e.d.

We now assume the fibers in X are autoparallel with respect to the given connection [11], i.e., we need to assume the restriction of the connection to a fiber yields a connection. For a vector $V \in TF_x$, let P_V denote the parallel translation along the geodesic $\exp(tV)$, $t \in [0, 1]$, provided that the geodesic is defined for all such t. Also, recall that $\chi^+(a, Z) = \lim_{n\to\infty} \frac{1}{n} \log \|Ta_x^n(Z)\|$.

Lemma 2.4. *There exist constants C and K such that if $V \in TF_x$ and $\|V\| < K$, then $\|P_V\| < C$.*

Proof. Let $l(t)$ be a geodesic in F starting at x. By continuity of the connection, for any $V \in TF_x$, $P_{l(t)}(V)$ is a continuous function in t. Hence, by continuity of the norm on F, $f(t) = \|P_{l(t)}(V)\|$ is also continuous.

For a fixed $C > 1$, continuity of f implies for every $x \in F$, there exists a neighborhood U_x of x such that for any geodesic generated by

$V \in TF_x$, starting at x and staying in U_x, we have $\|P_V\| < C$. Using the local diffeomorphic property of exp, there exists K_x such that for every $V \in TF_x$ with $\|V\| < K_x$, $\exp(tV)$ lies in U_x for $t \in [0, 1]$. For such V, we thus have $\|P_V\| < C$. Note that K_x varies continuously with x. Using compactness of F we can choose a $K > 0$ such that $K \leq K_x$ for every $x \in F$. The lemma now follows. q.e.d.

By the usual arguments, the previous result is independent of our choice of norm.

Proposition 2.5. *Fix $a \in A$ and suppose $x, y \in \Lambda$ and $l(t) = \exp(tZ)$ for $t \in [0, 1]$ is a geodesic from x to y with $\chi^+(a, Z) < 0$.*

(i) *If $V \in F_j(x)$, then $\chi_j(a, P_l(V)) \leq \chi_j(a, V)$.*
(ii) *If $V \in F_j(x)$ where $\chi_j(a, V)$ is minimal, then $P_l(V) \in F_j(y)$, i.e., P_l preserves the maximal contracting direction.*
(iii) *If $\chi_1(a) < \chi_2(a) < \cdots < \chi_f(a)$, then P_l preserves the flag $F_1 \subset F_1 \oplus F_2 \subset \cdots \subset F_1 \oplus \cdots \oplus F_f$.*

Proof. Since the G action preserves the connection, we have

$$Ta_y \circ P_l = P_{a \circ l} \circ Ta_x.$$

Note that $a^n \circ l(t) = a^n \circ \exp(tZ)$ is a geodesic at $a^n(x)$ in the direction $Ta^n_x(Z)$. Since $\chi^+(a, Z) < 0$, $\{\|Ta^n_x(Z)\|\}_{n \geq 0}$ is bounded, and, in fact, converges to 0. Thus, there exists N such that for every $n \geq N$, $\|Ta^n_x(Z)\| < K$, where K is as in the previous lemma. Therefore, there exists $C > 0$ such that

$$\|Ta^n_y \circ P_l(V)\| \leq C\|Ta^n_x(V)\| \qquad \forall n \geq 0.$$

Thus,

$$\lim_{n \to \infty} \frac{1}{n} \log \|Ta^n_y \circ P_l(V)\| \leq \lim_{n \to \infty} \frac{1}{n} \log(C\|Ta^n_x(V)\|) = \chi_j(a).$$

This proves the first claim. The second statement follows as minimality of χ_j assures us that equality is achieved, and that $P_l(V) \in F_j(y)$. The third claim is deduced immediately from the first. q.e.d.

Remark 2.4. Without mention, we have made substantial use of our assumption that the fibers are autoparallel. This assumption assures us that parallel translation along a path in the fiber takes tangent vectors to the fiber to tangent vectors to the fiber.

If $l(t) = \exp tZ$ is a geodesic and there exists $a \in A$ such that $\chi^+(a, Z) < 0$, then call l a *contracting geodesic* for a.

Using Fubini's theorem, there exists $x \in \Lambda$ such that $\Lambda \cap F_x$ is conull in F. We now fix such an x.

Proposition 2.6. *Let R and T be the curvature and torsion tensors of the connection on F_x, and let $X_i \in F_i(x)$. Then*

(i) (a) $R(X_1, X_2)X_3 = 0$, *or*
   (b) $R(X_1, X_2)X_3$ *has Lyapunov exponent* $\chi_1 + \chi_2 + \chi_3$.
(ii) (a) $T(X_1, X_2) = 0$, *or*
   (b) $T(X_1, X_2)$ *has Lyapunov exponent* $\chi_1 + \chi_2$.

Proof. Let $W = R(X_1, X_2)X_3$. We have

$$\|W\| \leq \|R\| \cdot \prod_i \|X_i\|.$$

Hence,

$$\begin{aligned}\log(\|Ta_x^n(W)\|) &= \log(\|(R(Ta_x^n X_1, Ta_x^n X_2))(Ta_x^n X_3)\|) \\ &\leq \log(\|R\| \prod_i \|Ta_x^n X_i\|) \\ &= \log \|R\| + \sum_i \log \|Ta_x^n X_i\|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n} \log \|Ta_x^n(W)\| \leq \sum_i \chi_i(a). \tag{1}$$

Replacing a with $b = a^{-1}$, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \|Tb_x^n(W)\| \leq \sum_i \chi_i(b) = -\sum_i \chi_i(a), \tag{2}$$

and if $W \neq 0$, then

$$\lim_{n\to-\infty} \frac{1}{n} \log \|Ta_x^n(W)\| \geq \sum_i \chi_i(a). \tag{3}$$

Combining (1) and (3) and using regularity of $x \in \Lambda$ yield

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Ta_x^n(W)\| = \sum_i \chi_i(a).$$

A similar argument works for T as well.

2.2. Construction of C^r sections. Throughout this subsection we assume that the fibers F in the bundle X over Y are f dimensional ($f \geq 3$), and that the algebraic hull $L = SL(f, \mathbb{R})$. Using superrigidity, by passing to a finite (algebraic) cover of G, we may assume $\alpha' \sim \pi : G \to L$ is a surjection. If D is the set of diagonal matrices in $SL(f, \mathbb{R})$, then we may choose $A \subset G$ such that $\pi(A) = D$. Thus the eigenvalues for $\pi(a)$ are just the elements along the diagonal. By Proposition 2.3, we have

Lemma 2.7. *Let $\{\chi_i\}$ be the set of Lyapunov exponents corresponding to the A action. Then the following hold:*

(i) *If $x \in \Lambda$ and $TF_x = \bigoplus F_i(x)$ is the Oseledec decomposition corresponding to the A action, then $\dim(F_i(x)) = 1$ for all i.*
(ii) *If σ is a permutation on the set of f elements, then there exists $a \in A$ such that*

$$\chi_{\sigma(1)}(a) < \chi_{\sigma(2)}(a) < \cdots < 0 < \chi_{\sigma(f)}(a).$$

Recall we have $x \in X$ such that $F_x \cap \Lambda$ is conull in F_x. Since the fibers are autoparallel, the connection on X yields one on F_x. Let R and T be the curvature and torsion of this connection.

Proposition 2.8. $R \equiv 0$ *and* $T \equiv 0$.

Proof. Let $i, j, k \in \{1, 2, \cdots, f\}$. If $R(X_i, X_j)X_k \neq 0$, then $\chi_i + \chi_j + \chi_k$ is a Lyapunov exponent, and hence, must be χ_l for some $l \in \{1, 2, \cdots, f\}$. Using Lemma 2.3, this corresponds to an algebraic relation among the a_i's for every $\pi(a) \in D$. In other words, we have that $a_i a_j a_k = a_l$ for all diagonal matrices in $SL(f)$. However, no such relation exists for *all* matrices in D. Thus, $R(X_i, X_j)X_k = 0$ for all i, j, k. Hence, $R \equiv 0$. A similar argument shows $T \equiv 0$. q.e.d.

If $x \in \Lambda$ and $TF_x = \bigoplus_{i=1}^{f} F_i(x)$, let $W_j(x) = \bigoplus_{i \neq j} F_i(x)$ be a hyperplane in TF_x (so $W_j(x)$ has constant j-coordinate), and let $V_j(x) = \exp B_j(x)$ where exp is defined. Since $R, T \equiv 0$, in some neighborhood U of x we may assume that U is a flat affine space, and that exp is the identity map. For $x \in \Lambda$, let $\{X_i(x)\}$ be a basis of TF_x with $X_i(x) \in F_i(x)$.

Lemma 2.9. *Let $x \in \Lambda \cap U$ and suppose $z \in V_i(x) \cap \Lambda \cap U$. Then $F_j(x)$ is parallel to $F_j(z)$ for $j \neq i$.*

Proof. Pick $a \in A$ such that $a_i > 1$ and a_j is minimal. Then any vector in $W_i(x)$ is contracting for a, and since a_j is minimal, parallel translation along any geodesic from x to z in $V_i(x)$ maps $F_j(x)$ to $F_j(z)$. However, the parallel translation in U is just ordinary translation, and so $F_j(x)$ is parallel to $F_j(z)$.

Lemma 2.10. *Suppose $l(t) = \exp tX_i(x)$ is a geodesic in U from x through $z \in \Lambda$. Then $F_j(x)$ is parallel to $F_j(z)$ for all $j = 1, 2, \cdots, f$.*

Proof. Pick $k \neq i$. Then $l(t)$ lies in $V_k(x)$, so by Lemma 2.9, $F_j(x)$ is parallel to $F_j(z)$ for all $j \neq k$. It remains only to see $F_k(x)$ is parallel to $F_k(z)$. However, we repeat the same argument using $l \neq i, k$ (possible since F is at least 3 dimensional), and conclude $F_k(x)$ is parallel to $F_k(z)$. q.e.d.

We define a geodesic of the form $l(t) = \exp tX_i(x)$ for $x \in \Lambda$ to be a *primary geodesic.*

Remark 2.5. The upshot of this lemma is that if we can join two regular points by a primary geodesic, then the entire decomposition is preserved. If we can connect enough of the regular points together in this fashion (enough, of course, meaning a conull set), we will have established that the Oseledec decomposition is preserved by parallel translation. From there, we simply define, via parallel translation, a decomposition on all of F_x, which is consistent with the Oseledec decomposition. Of course, we need to demonstrate that we can join enough regular points via these primary geodesics.

2.2.1. Ultraregular points. We now wish to establish the existence of a conull set of regular points with some special properties.

Proposition 2.11. *There exists a conull set* $\Lambda_0 \subset \Lambda$ *such that if* $x \in \Lambda_0$, *then almost every point of* $\exp tX_i(x)$ *lies in* Λ_0 *for all* $i = 1, 2, \cdots, f$ *and all* $t \in \mathbb{R}$.

We begin the proof by noting that we need only show the proposition holds in a neighborhood of every point. Throughout this section, then, we fix an $x \in F_x$, and a neighborhood U of x which we may assume is a flat affine space.

Lemma 2.12. *Suppose* $x, y \in U \cap \Lambda$. *If* $V_i(x) \cap V_i(y) \cap U \neq \varnothing$, *then* $V_i(x) = V_i(y)$ *for any* $i = 1, 2, \cdots, f$.

Proof. Pick $a \in A$ such that $a_i < 1$. Let $z \in V_i(x) \cap V_i(y) \cap U$. Then there exist a geodesic $l_1(t) = \exp(tZ_1)$ with $Z_1 \in W_i(x)$ joining x to z, a geodesic $l_2(t) = \exp tZ_2$ with $Z_2 \in W_i(y)$ joining y to z, and a geodesic $l_3(t) = \exp tZ_3$ joining x to y. Since Z_1 and Z_2 lie in $W_i(x)$ and $W_i(y)$ respectively, both l_i's are contracting for a. Hence $\|Ta_x^n(Z_1)\|, \|Ta_y^n(Z_2)\| \to 0$ exponentially as $n \to \infty$. By the triangle inequality, $\|Ta_x^n(Z_3)\| \to 0$ exponentially as $n \to \infty$. Since $W_i(x)$ consists of vectors with exactly this property, $Z_3 \in W_i(x)$, and therefore, $V_i(x) = V_i(y)$. q.e.d.

A Lipschitz Function. We now wish to view $V_1 : U \cap \Lambda \to \{$Hyperplanes in $U\}$. Choose a fixed $x_0 \in U \cap \Lambda$ and a basis for U at x_0. Use this basis to define a norm, so that for $r > 0$, $B_r(x) = \{y \mid \|x - y\| < r\}$. Then there exists $a > 0$ such that $B_a(x_0) \subset U$. Let $U_0 = B_{a/4}(x_0)$.

Let $L = \exp tX_1(x_0)$. Then L intersects $V_1(x)$ transversally for all $x \in B_a(x_0)$; if not, then $L \subset V_1(x)$ and therefore $x_0 \in V_1(x) \cap V_1(x_0)$ contradicting the previous lemma. For all $x \in B_a(x_0)$, define $G(x)$ to be the intersection of $V_1(x)$ with L, and let $\Theta(x)$ be the angle between L and the plane $V_1(x)$. Since L is transversal to $V_1(x)$, $\Theta(x) > 0$.

Lemma 2.13. *There exists* $\theta > 0$ *such that* $\Theta(x) \geq \theta$ *for all* $x \in U_0 \cap \Lambda$.

Proof. Choose $x_n, x \in U_0 \cap \Lambda$ such that $x_n \to x$. If $\Theta(x_n) \not\to \Theta(x)$, then there exist $\epsilon > 0$ and a subsequence x_{n_k} such that $x_{n_k} \to x$, but $|\Theta(x_{n_k}) - \Theta(x)| \geq \epsilon$. Thus for large n_k, $V_1(x_{n_k}) \cap V_1(x) \cap U \neq \varnothing$, contradicting the previous lemma. So $\Theta(x)$ is continuous on $U_0 \cap \Lambda$, and therefore, for $\theta > 0$, $\Theta^{-1}(\theta, \pi/2]$ is an open set in $U_0 \cap \Lambda$. By shrinking U_0 if necessary, the lemma follows.

Proposition 2.14. $G : U_0 \cap \Lambda \to L$ *is Lipschitz. More specifically,*

$$|G(x) - G(y)| \leq \frac{7}{\sin\theta}\|x - y\|.$$

Proof. Let $x, y \in U_0 \cap \Lambda$ so that $\|x - y\| = \epsilon < a/2$. If $x, y \in L$, then $|G(x) - G(y)| = \|x - y\|$. So, we assume that $x \notin L$. Let P be the plane containing L and x, and for every z, let $l(z)$ be the line of intersection of $V_1(z)$ and P. Note, then, that $\Theta(x)$ is less than the angle between L and $l(x)$ in P. Let $B_0 = U_0 \cap P$, a circle with radius $a/4$, since $L \subset P$, and L is a diameter of U_0.

Since $V_1(x)$ and $V_1(y)$ do not intersect in U, $l(x)$ and $l(y)$ do not intersect in B_0. Let R be the intersection of the line perpendicular to $l(y)$ through x, and let $r = |x - R|$. So $r < \epsilon$.

Assume that all lines are defined on all of $\mathbb{R}^n$. If $l(x)$ and $l(y)$ never intersect, then

$$|G(x) - G(y)| \leq \frac{r}{\sin\Theta(x)} < \frac{\epsilon}{\sin\theta} = \frac{\|x - y\|}{\sin\theta}.$$

So assume $l(x)$ and $l(y)$ intersect at the point S. Let C be the point of intersection of $l(x)$ and the line perpendicular to $l(y)$ through $G(y)$, and let $\gamma = |C - G(y)|$. Also, let D be the point on $l(x)$ closest to $G(y)$ and let $\delta = |D - G(y)|$.

Exploiting the similarity of the triangles $SCG(y)$ and SxR yields

$$\delta < \gamma = \frac{r|S - G(y)|}{|S - R|} \leq \frac{\epsilon(|S - R| + |R - G(y)|)}{|S - R|} = \epsilon\left(1 + \frac{|R - G(y)|}{|S - R|}\right).$$

Now,

$$|R - G(y)| \le |R - x| + \|x - y\| + \|y - G(y)\| \le a/2 + a/2 + 2a = 3a$$

by Lemma 2.15 below. Since $R \in B_{a/4}(x_0)$ and $S \notin B_a(x_0)$, $|S - R| > a/2$. Thus,

$$\delta < \epsilon\left(1 + \frac{|R - G(y)|}{|S - R|}\right) < \epsilon\left(1 + \frac{3a}{|S - R|}\right) < \epsilon\left(1 + \frac{3a}{a/2}\right) = 7\epsilon.$$

Finally, using the right triangle $G(x)DG(y)$ we have

$$|G(x) - G(y)| \le \frac{\delta}{\sin\Theta(x)} \le \frac{7\epsilon}{\sin\theta} = \frac{7}{\sin\theta}\|x - y\|,$$

as claimed. q.e.d.

The proof shows that G is uniformly continuous on $U_0 \cap \Lambda$. Hence, we can extend G to a continuous and Lipschitz function on all of U_0.

Lemma 2.15. *In the notation from Proposition* 2.14, $|x - G(x)| < 2a$ *for all* $x \in U_0 \cap \Lambda$.

Proof. If $l(x)$ and $l(x_0)$ do not intersect in P, then $l(x)$ and $l(x_0)$ are parallel, and since L is perpendicular to $V_1(x_0)$, the closest point on $l(x)$ to $l(x_0)$ is $G(x)$. But $|x - x_0| < a/4$, so $|x_0 - G(x)| < a/4$, and therefore $|x - G(x)| \le |x - x_0| + |x_0 - G(x)| \le a/2$.

Suppose now that $l(x)$ and $l(x_0)$ intersect at S. Then $|x_0 - S| > a$. Let k be the line tangent to B_0 through S such that if K is the intersection of k and L, then $G(x)$ lies on L between x_0 and K. Let J be the intersection of k and B_0. Since x_0 is the center of B_0 and k is tangent to B_0, the triangles x_0JK and x_0JS are both right triangles. Note $|x_0 - J| = a/4$. Hence,

$$\frac{|J - x_0|}{|x_0 - S|} = \frac{a/4}{|x_0 - S|} < \frac{a/4}{a} = \frac{1}{4}.$$

Thus,

$$\frac{|J - S|}{|x_0 - J|} = \sqrt{1 - \left(\frac{|x_0 - J|}{|X_0 - S|}\right)^2} > \frac{\sqrt{15}}{4} > \frac{3}{4}.$$

By similar triangles, we have

$$\frac{|J - K|}{|x_0 - J|} = \frac{|x_0 - J|}{|J - S|} \le \frac{1/4}{3/4} = \frac{1}{3}.$$

But,

$$|x_0 - K| \le |x_0 - J| + |J - K| \le |x_0 - J| + \frac{|x_0 - J|}{3} < a,$$

and as $G(x)$ lies on L between x_0 and K, this implies $|x_0 - G(x)| < a$, from which we conclude

$$|x - G(x)| \leq |x - x_0| + |x_0 - G(x)| < a/4 + a < 2a,$$

which proves the lemma.

Theorem 2.16 (A type of Fubini theorem). *Consider a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$. If U is a Lebesgue measurable set, then*

$$\int_U J_n f(x)\, d\mathscr{L}^m x = \int_{\mathbb{R}^n} \mathscr{H}^{m-n}(U \cap f^{-1}\{y\})\, d\mathscr{L}^n y,$$

where $\mathscr{L}^n$ is n-dimensional Lebesgue measure, $\mathscr{H}^m$ is m-dimensional Hausdorff measure, and $J_n f(x) = \| \bigwedge_n Df(x)\|$ is the n-dimensional Jacobian of f at x (this is well defined as Lipschitz functions are differentiable almost everywhere [4, 3.1.6]).

Proof. This is 3.2.11 from [4]. q.e.d.

By applying this theorem, we see that a type of Fubini's Theorem result holds, and we may conclude that almost every point on almost every $V_1(y) \cap U_0$ lies in Λ.

Lemma 2.17. *Let $S \subset \{1, 2, \cdots, f\}$, $W_S(x) = \bigcap_{i \in S} W_i(x)$, and $V_S(x) = \exp W_S(x)$. Then the following hold:*

(i) *There exists a conull dense $\Lambda_i \subset \Lambda$ such that if $x \in \Lambda_i$ then almost every point in $V_i(x)$ lies in Λ_i.*

(ii) *There exists a conull set $\Lambda_S \subset \Lambda$ such that if $x \in \Lambda_S$ then almost every point of $V_S(x)$ lies in Λ_S.*

Proof. The first statement has been proven above. To see the second statement, assume that $S = \{i, j\}$. Note that each $V_i(x)$ is a hyperplane, and that by Lemma 2.9, $V_i(x) \cap V_j(x)$ are parallel hyperplanes of codimension 2 for all x. Hence, the usual Fubini's Theorem applies. Similarly, for general S, the $F_S(x)$ are also hyperplanes of higher codimension, and so the usual Fubini's Theorem still holds.

Proof of Proposition 2.11. Let $\Lambda_0 = \bigcap_{S_j} \Lambda_{S_j}$ where

$$S_j = \{1, 2, \cdots, f\} \setminus \{j\}.$$

q.e.d.

Proposition 2.18. *Fix $x \in \Lambda_0$. Then there exists a conull set $\Lambda_x \subset \Lambda_0$ such that for every $y \in \Lambda_x$ there is a path consisting of broken primary geodesics from x to y.*

Proof. Pick a neighborhood U of x, which we assume is a flat affine space. Since $x \in \Lambda_0$, the set $\Gamma_1 = \{\exp tX_1(x)\}_{t \in \mathbb{R}} \cap \Lambda_0$ is conull in

$\{\exp tX_1(x)\}_{t\in\mathbb{R}}$. By Lemma 2.10, if $y \in \Gamma_1$, then $X_i(x)$ is parallel to $X_i(y)$ for all i, and therefore, $\exp tX_2(y)$ lies in the plane $\bigcap_{j=3}^{f} V_j(x)$ for all $y \in \Gamma_1$, $t \in \mathbb{R}$. By Fubini's Theorem, the set

$$\Gamma_2 = \{\{\exp tX_2(y)\}_{t\in\mathbb{R}} \cap U\}_{y\in\Gamma_1}$$

is conull in $\bigcap_{j=3}^{f} V_j(x) \cap U$. Further, for every $z \in \Gamma_2$, $X_i(x)$ is parallel to $X_i(z)$ for all i. By Fubini again,

$$\Gamma_3 = \{\{\exp tX_3(z)\}_{t\in\mathbb{R}} \cap U\}_{z\in\Gamma_2}$$

is conull in $\bigcap_{j=4}^{f} V_j(x) \cap U$. We can continue constructing these Γ_i's for all $i = 1, 2, \cdots, f$. Let $\Lambda_x = \Gamma_f$. Then Λ_x is conull in U, and by construction, there exists a broken primary geodesic from x to y for any $y \in \Lambda_x$.

If V is another neighborhood of a flat affine space which intersects U, then there exists $y \in \Lambda_x \cap V$. Repeating the above argument for y, we may conclude that Λ_x is conull in $U \cup V$. The proposition now follows since we can cover F_x with a finite number of such neighborhoods.

2.2.2. Regularity of the Oseledec decomposition.

Theorem 2.19. *Let X be a compact fiber bundle over Y with fibers F. Let G be a connected semisimple Lie group of higher rank without compact factors. Suppose G acts ergodically on X via bundle automorphisms preserving a volume density and a C^r connection such that the fibers are autoparallel. If the algebraic hull of $\alpha': G \times X \to SL(f)$, the derivative cocycle in the fiber direction, equals $SL(f)$, then, by possibly having to pass to a finite algebraic cover of G, there exists an abelian subgroup $A \subset G$ such that the Oseledec decomposition of TF corresponding to A is parallel, C^r regular, and everywhere defined on F_x for almost every $x \in X$. In particular, there exists a C^r section of the full flag bundle over $T(F_x)$.*

Proof. We follow the outline described in §2.2. By Proposition 2.18, there exists a conull set of regular points such that any two may be connected by a series of primary geodesics. By Lemma 2.10, the Oseledec decomposition is preserved by parallel translation along this path. Note that any path is homotopic to a path consisting of a series of primary geodesics. (This is certainly true locally, so we need only cover our (compact) path with a finite number of neighborhoods.) Since $R \equiv 0$, parallel translation along homotopic paths is identical. Hence, the Oseledec decomposition is preserved by parallel translation along any path. Define the decomposition to be parallel translation to points in the complement of Λ_x. By construction, the decomposition is parallel. q.e.d.

A similar result holds for certain actions of lattices on manifolds.

Theorem 2.20. *Let G be as in Theorem 2.19. Suppose $\Gamma \subset G$ is a lattice acting on an f-dimensional manifold M preserving a volume density and a C^r connection. If the algebraic hull of the derivative cocycle $\alpha: \Gamma \times M \to SL(f)$ equals $SL(f)$, and if $\pi(\Gamma) \subset SL(f)$ contains a lattice, where $\pi : G \to SL(f)$ is the superrigidity homomorphism, then, by possibly having to pass to a finte cover of Γ, the Oseledec decomposition on M corresponding to an abelian subgroup $A \subset \Gamma$ is parallel, C^r regular, and everywhere defined. So, there exists a C^r section of the full flag bundle over M.*

Proof. The proof follows exactly as in Theorem 2.19 once we have established the existence of a suitable abelian subgroup, i.e., we need a result comparable to Lemma 2.7. Since $\pi(\Gamma)$ contains a lattice in $SL(f)$, using Lemma 3.2 in [6] and its preceding remarks, we complete the proof.

3. $C^{(r,s)}$ Superrigidity

We will need to prove a generalization of Zimmer's topological superrigidity for the sequel. The difference between the results here and those presented in [14] lies in the regularity of the sections under consideration. In [14], Zimmer proves a result analogous to Theorem 3.4 for C^r sections. The point here is to strengthen the geometric meaning of the previously constructed sections, and these sections are not C^r regular.

3.1. Preliminaries. Let G be a semisimple Lie group acting on set S. Then $s \in S$ is called a *parabolic invariant* if there exists a parabolic subgroup $Q \subset G$ such that Q fixes s. In particular, if G acts by automorphisms of a principal H-bundle $P \to M$, and V is an H-space with $E_V \to M$ the associated bundle, then a *parabolic invariant section* is a section of E_V invariant under a parabolic subgroup of G.

Let $P \to M$ be a principal H-bundle on which G acts via bundle automorphisms. If $\pi: G \to H$ is a homomorphism, then a section $s: M \to P$ is said to be *totally π-simple* if for $g \in G$ and $m \in M$,

$$s(gm) = g.s(m).\pi(g)^{-1}.$$

Here, of course, M and P are right G-spaces, and P is a left H-space.

Let X be a fiber bundle over Y with fibers F. For any function $\phi: X \to M$, and any neighborhood $U \subset Y$ trivializing X, we have $\phi|_{p^{-1}(U)}: F \times U \to M$. Call $\phi : X \to M$ a $C^{(r,s)}$ *function* if for any such U, $\phi|_{p^{-1}(U)}$ is C^r as a function of F and C^s as a function of U.

Similarly, if $P \to X$ is a principal H-bundle over X, V is an H-space, and $\Phi : P \to V$ is an H-equivariant map, call Φ an *H-equivariant $C^{(r,s)}$ function* if locally Φ is a C^r function of F and a C^s function of Y. Note that by H-equivariance, Φ is necessarily C^∞ as a function of H.

Although most of what follows will be valid for most (r, s), the most interesting results follow when either r or s is measurable. Of particular interest to us is the case where s is measurable. Let $C^{(r,s)}(X; E_V)$ be the set of $C^{(r,s)}$ sections of the associated bundle E_V over X.

Proposition 3.1. *There exists a natural bijective correspondence between $C^{(r,s)}(X; E_V)$ and H-equivariant $C^{(r,s)}$ maps $P \to V$. Furthermore, G-invariant elements of $C^{(r,s)}(X; E_V)$ correspond to H-equivariant G-invariant $C^{(r,s)}$ maps $P \to V$.*

Proof. The proof is standard noting the appropriate changes in regularity. See [1], for example. q.e.d.

Let $r \in [0, \infty]$ and s be measurable. Call a set $U \subset X$ a *$C^{(r,s)}$ generic* set if U is conull in X and for almost every $x \in U$, $F_x \cap U$ is open, dense in F_x, where F_x is the fiber in X through the point x.

Now suppose G acts by automorphisms of a principal H-bundle $P \to X$ where H is an algebraic group. Assume that the G action on X is ergodic. An algebraic subgroup $L \subset H$ is called a *$C^{(r,s)}$ algebraic hull* for the G action on P if

(i) there exist a G-invariant $C^{(r,s)}$ generic set $U \subset X$ and a G-invariant $C^{(r,s)}$ section of $E_{H/L}|_U \to U$, and
(ii) the first assertion is false for any proper algebraic subgroup of L.

Proposition 3.2. *Assume the situation described in the previous definition. Then the following hold*:

(i) *$C^{(r,s)}$ algebraic hulls always exist.*
(ii) *Any two $C^{(r,s)}$ algebraic hulls are conjugate in H.*
(iii) *If there exist a G-invariant $C^{(r,s)}$-generic $U \subset X$ and a $C^{(r,s)}$ section of $E_{H/L_1}|_U \to U$, then L_1 contains a $C^{(r,s)}$ hull.*

The first and last statements follow from the descending chain condition on algebraic subgroups. The second statement will require

Lemma 3.3. *Let G be ergodic on X, and $P \to X$ a principal H-bundle on which G acts by bundle automorphisms. Let H be a real algebraic group and let V be a quasi-algebraic H-space. Let ϕ be a G-invariant $C^{(r,s)}$ section of $E_V \to X$, with $\Phi : P \to V$ the corresponding H-equivariant G-invariant $C^{(r,s)}$ map. Then there exist a G-invariant $C^{(r,s)}$ generic $U \subset X$ and an H-orbit $\mathcal{O} \subset V$ such that $\phi(P|_U) \subset \mathcal{O}$.*

Proof. Since $\Phi : P \to V$ is H-equivariant, factoring by the H action, we obtain $\tilde{\Phi} : X \to H\backslash V$. The H action on V is quasi-algebraic and therefore tame. As the G action on X is ergodic, $\tilde{\Phi}$ is constant on $U \subset X$ a conull set. By Fubini, for almost every $x \in U$, $F_x \cap U$ is conull in F_x. But then $\tilde{\Phi}|_{F_x}$ is a C^r function constant on the conull set $F_x \cap U \subset F_x$. Hence $F_x \cap U = F_x$ for such x, and U is $C^{(r,s)}$ generic.

Proof of Proposition 3.2. Let $U_i \subset X$ be two $C^{(r,s)}$ generic G-invariant sets with $L_i \subset H$ algebraic subgroups and $\Phi_i : P|_{U_i} \to H|_{L_i}$ G-invariant H-equivariant $C^{(r,s)}$ maps. Let $\Phi = (\Phi_1, \Phi_2) : P|_{U_1 \cap U_2} \to H/L_1 \times H/L_2$. (Note that $U_1 \cap U_2$ is a $C^{(r,s)}$ generic set.) Apply Lemma 3.3 to $U_1 \cap U_2$ to obtain a $C^{(r,s)}$ generic set $U \subset U_1 \cap U_2$ and a single H orbit $\mathscr{O} \subset H/L_1 \times H/L_2$ such that $\Phi(P|_U) \subset \mathscr{O}$. Note that the stabilizer of a point in $H/L_1 \times H/L_2$ under the H action takes the form $h_1 L_1 h_1^{-1} \cap h_2 L_2 h_2^{-1}$ for some $h_1, h_2 \in H$. Thus, we have $\Phi(P|_U) \to H|_{h_1 L_1 h_1^{-1} \cap h_2 L_2 h_2^{-1}}$. By definition of $C^{(r,s)}$ hull, $h_1 L_1 h_1^{-1} \cap h_2 L_2 h_2^{-1}$ contains L_1 so that L_1 is contained in a conjugate of L_2. Similarly, L_2 is contained in a conjugate of L_1. Since the L_i's are algebraic subgroups, L_1 and L_2 are actually conjugate. q.e.d.

Let G act via principal bundle automorphisms on $P(X, H)$ with $C^{(r,s)}$ algebraic hull L. The G action is $C^{(r,s)}$ *complete* if

(i) there exists a $C^{(r,s)}$ generic set U such that for almost every $x \in U$, $F_x \subset U$, and

(ii) there exists a $C^{(r,s)}$ section $E_{H/L}|_U \to U$.

Given a principal H-bundle $P \to X$, and V an H-space, a section ϕ of E_V is said to be *effective* for P if H acts effectively on $\Phi(P)$ where $\Phi : P \to V$ is the corresponding H-map to ϕ. Suppose G acts ergodically and $C^{(r,s)}$ completely on P where H is an algebraic group and V is an algebraic variety. Then a $C^{(r,s)}$ section $\phi : X \to E_V$ is $C^{(r,s)}$-*G-effective* if it is effective for $P_1 \subset P$ where P_1 is a G-invariant reduction to $L \subset H$, the $C^{(r,s)}$ algebraic hull.

We are now ready to state the main result of this section.

Theorem 3.4 ($C^{(r,s)}$ superrigidity). *Let G be a connected semisimple Lie group, $\mathbb{R}$-rank$(G) \geq 2$, with G acting via bundle automorphisms on $P(X, H)$, H an algebraic $\mathbb{R}$-group, V an $\mathbb{R}$-variety on which H acts algebraically, and G acting ergodically on X with respect to a probability measure μ where* $\mathrm{supp}(\mu) = X$. *Assume the action is $C^{(r,s)}$ complete. If* *there exists a G effective $C^{(r,s)}$ parabolic invariant section ψ of $E_V \to X$, then, by possibly passing to a finite cover of G, there exist*

(i) *a homomorphism $\pi : G \to H$,*
(ii) *$v_0 \in V$, and*
(iii) *a totally π-simple $C^{(r,s)}$ section s of $P \to X$*

such that ψ is the associated section (s, v_0), i.e., $\psi(x) = [s(x), v_0]$.

Remark 3.1. (i) We can dispense with the completeness assumption; however, the totally π-simple $C^{(r,s)}$ section s will be defined only where ψ is defined. So, in essence, the necessity of the completeness assumption is simply to ensure that the section ψ exists.

(ii) Effectiveness of ψ will ensure that we can see enough of the $C^{(r,s)}$ algebraic hull by looking at V. This is to avoid the obvious degeneracy problems, e.g., if the $C^{(r,s)}$ algebraic hull were in the kernel of the H action on V. By passing to a suitable subquotient, we can still obtain results without the effectiveness assumption. Let $P_1 \subset P$ be the G-invariant reduction to the $C^{(r,s)}$ algebraic hull L, and let $N \subset L$ be the maximal normal subgroup of L pointwise fixing $\Phi(P_1)$. We can then obtain a $C^{(r,s)}$-G-effective section of the V associated bundle to the principal bundle P_1/N, and apply the theorem to this new situation.

(iii) The hypothesis concerning the parabolic invariant section is the key point in extending measurable superrigidity to topological and $C^{(r,s)}$ superrigidity. The proofs of all versions of superrigidity depend on the existence of parabolic invariance; however, in the measurable case, it is always possible to deduce the existence of relevant parabolic invariant sections. This is not the case in the situations of higher regularity, thus the need for this additional assumption.

3.2. Proof of $C^{(r,s)}$ superrigidity. By $C^{(r,s)}$ completeness, we may assume H is the $C^{(r,s)}$ algebraic hull. Let Q be the parabolic subgroup such that ψ is Q-invariant. Let $B \subset Q$ be a minimal parabolic subgroup. By Proposition 3.1, ψ corresponds to a Q-invariant, H-equivariant $C^{(r,s)}$ map $\Psi : P \to V$. If $\Psi_0 : X \to H\backslash V$ is the induced map on H orbits, then Ψ_0 is B-invariant. Moore's Theorem implies B is ergodic on X; therefore, by Lemma 3.3, there exist a $C^{(r,s)}$ generic set $X_0 \subset X$ and an H orbit $V_0 \subset V$ such that $\Psi_0(x) \in V_0$ for every $x \in X_0$. We may assume $\overline{V_0} = V$.

Let L be the measurable algebraic hull of the G action on P. Let R be the measurable G-invariant subbundle which is a measurable principal L bundle. Then for almost every $q \in R$, $\Psi_0(q)$ lies in a single L orbit, say $W_0 \subset V$. Note that $W_0 \subset V_0$ and $H.W_0 = V_0$.

Now define $F : P \times G/B \to V$ so that $F(p, g) = \Psi(g^{-1}p)$. Since Ψ is B-invariant, F is indeed well defined. From the proof of measurable superrigidity [15], we have

Proposition 3.5. *By passing to a finite (algebraic) cover of* G,

(i) *there exists a regular homomorphism* $\sigma : G \to L$, *and*

(ii) *for almost every* $r \in R$, *there exists a conjugate* σ^r *of* σ *in* L, *such that for almost all* $r \in R$, $F(r, g) = \sigma^r(g).\Psi(r)$. *Further, if* L_1 *is the stabilizer in* L *of* W_0, *then* $\sigma : G \to L/L_1$ *is a surjection.*

Next define $\Phi: P \to \text{Func}(G/B, V)$ as $\Phi(p)(g) = F(p, g) = \Psi(g^{-1}p)$. Proposition 3.5 implies that for almost every $q \in R$, $\Phi(q) : G/B \to W_0$ is a regular surjective map. As B is parabolic, G/B is a complete variety, and therefore W_0 is Zariski closed in V. We also have

Proposition 3.6. *There exists* $\Phi_0 \in \text{Reg}(G/B, W_0)$ *such that for almost every* $q \in R$, $\Phi(q) \in L.\Phi_0$.

Proof. Since Ψ is L-equivariant, so is the map $\Phi: R \to \text{Reg}(G/B, W_0)$. Let $\tilde{\Phi}: L\backslash R : X \to L\backslash \text{Reg}(G/B, W_0)$ be the map on L orbits. Note that $\tilde{\Phi}$ is B-invariant since Ψ is B-invariant. Since B is ergodic on X, $\tilde{\Phi}$ must be constant on a conull set. q.e.d.

Let $R_1 = \{q \in R \mid \Psi(q) \in W_0 \text{ and } \Phi(q) \in L.\Phi_0\}$. Then R_1 is conull in R, and if P_1 is the H saturation in P of R_1, then P_1 is conull in P and for every $p \in P_1$, $\Phi(p)$ is regular and $\Phi(p) \in H.\Phi_0 \subset \text{Reg}(G/B, V_0)$.

By Theorem 3.3.1 in [15], the H action on $\text{Reg}(G/B, V)$ is tame, so $H.\Phi_0$ is open in its closure in $C^{(r,s)}(G/B, V)$. If $P_2 = \{p \in P \mid \Phi(p) \in H.\Phi_0\}$, then since Φ is H-equivariant $C^{(r,s)}$ regular, P_2 is $C^{(r,s)}$ generic.

Let $\mathscr{S}(V)$ be the space of closed subvarieties of V. Define $\mathscr{X}: P_2 \to \mathscr{S}(V)$ so that $\mathscr{X}(p) = \Phi(p)(G/B)$. Then $\mathscr{X}$ is H-equivariant and partially G-invariant (i.e., if $p, gp \in P_2$, then $\mathscr{X}(p) = \mathscr{X}(gp)$). Also, $\mathscr{X}(P_2) \subset H.W_0 \subset \mathscr{S}(V)$. If H_1 is the stabilizer of W_0 in $\mathscr{S}(V)$, then we obtain an H-equivariant, partially G-invariant $C^{(r,s)}$ map $P_2 \to H/H_1$. This corresponds to a G-invariant $C^{(r,s)}$ section of $E_{H/H_1}|_U \to U$, where $U = G.p(P_2)$ ($p: P \to X$ is the standard projection). Since H is the $C^{(r,s)}$ algebraic hull for the G action on P, $H_1 = H$. So, H stabilizes W_0 and as W_0 is closed, we have

$$W_0 = V_0 = \overline{V_0} = V.$$

Thus, both H and L are transitive on V, and for every $p \in P_2$, $\Phi(p) : G/B \to V$ is a regular surjection.

To summarize, there is an H-equivariant $C^{(r,s)}$ map $\Phi : P_2 \to H.\Phi_0 \subset \mathrm{Reg}(G/B, V)$. However, we will need to know that $p(P_2)$ contains entire fibers, not just open dense subsets of fibers.

Lemma 3.7. *Let* $U = p(P_2)$. *If* $F_x \cap U$ *is conull in* F_x, *then* $F_x \subset U$.

Proof. Note that V is complete and a transitive H-space, and can therefore be embedded as a closed orbit in a projective space via a representation of H. Let $f_0 \in F_x$. Choose $\{p_n\}_{n\geq 1} \in P_2$ such that $p_n \to p_0$ and $p(p_i) = f_i$. Since Φ is continuous in the direction of F and H, and G/B is compact, $\Phi(p_n) \to \Phi(p_0)$ uniformly. As $\Phi(p_n) \in H.\Phi_0$, we can write $\Phi(p_n) = h_n.\Phi_0$ for $h_n \in H$. By Lemma 6.3 in [14], h_n is bounded in $PGL(n+1)$, and therefore converges to $h_0 \in PGL(n+1)$. Since H is algebraic, h_0 lies in the image of H in $PGL(n+l)$. Hence we conclude that $\Phi(p_0) = h_0.\Phi_0$. q.e.d.

Since Φ_0 is surjective, and H is effective on V, the stabilizer of Φ_0 in H is trivial, implying that Φ defines an H-equivariant $C^{(r,s)}$ map $P_2 \to H$ and therefore a $C^{(r,s)}$ section s of $P_2 \to U \subset X$ ($U = p(P_2)$) as follows: for $m \in U$, $s(m)$ is the element of P_m such that $\Phi(s(m)) = \Phi_0$. We have $\Phi(p)(ag) = \Psi(g^{-1}a^{-1}p) = \Phi(a^{-1}p)(g)$. Thus, for every $p \in P_2$, the G orbit of $\Phi(p)$ in $\mathrm{Reg}(G/B, V)$ is contained in $\Phi(P)$, and therefore in $H.\Phi_0$.

Using the proof of measurable superrigidity (or Lemma 3.5.2 from [15]), we conclude that for each $p \in P_2$, there exists a homomorphism $\pi_p : G \to H$ such that

$$\Phi(p)(g) = \pi_p(g).\Phi(p)(e).$$

Define $\pi_m = \pi_{s(m)}$. Then, for all $m \in U$, $g \in G$,

$$\Phi_0(g) = \pi_m(g).\Phi_0(e).$$

For any $a \in G$,

$$\Phi_0(ga) = \pi_m(ga).\Phi_0(e) = \pi_m(g)\pi_m(a).\Phi_0(e) = \pi_m(g).\Phi_0(a).$$

So, for any $g, a \in G$,

$$\pi_{m_1}\Phi_0(a) = \pi_{m_2}\Phi_0(a),$$

for any $m_1, m_2 \in U$. Again, since Φ_0 is surjective and H is effective on V, we have $\pi_{m_1}(g) = \pi_{m_2}(g)$ for all such m_1, m_2. Hence, there exists $\pi : G \to H$ such that $\pi = \pi_m$ for all $m \in U$.

As $\Phi: P_2 \to H.\Phi_0 \subset \mathrm{Reg}(G/B, V)$ is injective on the fibers of P, we need only show $\Phi(g.s(m).\pi(g)^{-1}) = \Phi(s(gm))$ to see that s is totally π-simple. But,

$$\begin{aligned}\Phi(g.s(m).\pi(g)^{-1})(a) &= \Psi(a^{-1}g.s(m).\pi(g)^{-1}) = \pi(g).\Psi(a^{-1}g.s(m)) \\ &= \pi(g).\Phi(s(m))(g^{-1}a) = \pi(g).\Phi_0(g^{-1}a) \\ &= \Phi_0(a) = \Phi(s(gm))(a).\end{aligned}$$

Finally, it is routine to verify that ψ is the section associated to s and $\Phi_0(e) \in V$.

4. Geometric implications

The purpose of this section is to explore some of the geometric consequences of our previous work.

4.1. Connection preserving actions on fiber bundles. Let X be a compact fiber bundle over Y with fibers F. Let G act ergodically on X via fiber bundle automorphisms preserving a smooth connection and a volume density, where G is a connected semisimple Lie group of higher rank. Also, assume that the fibers are autoparallel. Let $\beta : G \times Y \to \mathrm{Diff}(F)$, $\alpha : G \times X \to GL(x)$, and $\alpha' : G \times X \to GL(f)$ be the cocycles described in §2.1. Let L be the algebraic hull of α'.

Theorem 4.1. (i) *If L is compact, then there exists a smooth Riemannian metric g on F with respect to which F has a transitive group of isometries, i.e.,*

$$F \cong \frac{\mathrm{Isom}(F, g)}{\mathrm{Isom}(F, g)_x}.$$

(ii) *If $L = SL(f)$, where $f = \dim(F)$, then F is a torus.*

Remark 4.1. Since isometry groups of compact manifolds are rather special (particularly when they are large in comparison to the manifold), as are their closed subgroups, we have special restrictions on the possibilities for F. In low dimensions, these possibilities are few and easy to calculate. Examples not precluded by these results are also easy to construct.

4.1.1. *Proof of Theorem* 4.1.

L **Compact.** Compactness of L implies the existence of a measurable invariant metric in the direction of the fibers. Equivalently, there exists a β-invariant function $\Phi : Y \to \{\text{measurable Riemannian metrics on } F\}$.

Theorem 4.2. *There exists a smooth invariant metric on* F.

Proof. This is a main result of [2]. Note that this requires the fibers to be autoparallel. q.e.d.

Thus, we have a cocycle $\beta: G \times Y \to \mathrm{Diff}(G)$ and a β-invariant function $\Phi: Y \to \mathrm{Met}(F)$, the space of Riemannian metrics on F. Next, we employ

Theorem 4.3. *The* $\mathrm{Diff}(F)$ *action on* $\mathrm{Met}(F)$ *is tame.*

Proof. Theorem 7.4 of [3] gives a construction of a slice for the action of $\mathrm{Diff}(F)$ on $\mathrm{Met}(F)$. In particular, for every $m \in \mathrm{Met}(F)$, there exists a submanifold S_m of $\mathrm{Met}(F)$ such that there is a local cross section

$$\chi: \mathrm{Diff}(F)/\mathrm{Diff}(F)_m \to \mathrm{Diff}(F)$$

defined on a neighborhood U of the identity coset such that the map $U \times S_m \to \mathrm{Met}(F)$ is a homeomorphism onto a neighborhood of m, ($\mathrm{Diff}(F)_m$ being the stabilizer of m in $\mathrm{Diff}(F)$). This is enough to ensure that the action of $\mathrm{Diff}(F)$ on $\mathrm{Met}(F)$ is tame. See, for instance, 2.1.12 in [15]. q.e.d.

Applying the Cocycle Reduction Lemma 5.2.11 in [15], there exists $g \in \mathrm{Met}(F)$ such that β is equivalent to $\beta': G \times Y \to \mathrm{Diff}(F)_g = \mathrm{Isom}(F, g)$. As G acts ergodically on X, it must act ergodically on F (under the appropriate measurable trivialization of $X \cong Y \times F$); therefore F has an ergodic group of isometries. Compactness of F allows us to conclude that F therefore has a transitive group of isometries.

L **Noncompact.** Let $P \to X$ be the principal $SL(f)$ bundle over X. Let V be the space of full flags on $\mathbb{R}^f$. Theorem 2.19 establishes the existence of a section Φ of the associated bundle $E_V \to X$. Note that this section varies measurably in the Y direction on X, but is C^r regular in the direction of F. Also, note that this is a parabolic invariant section, with the parabolic subgroup being the stabilizer of the appropriate flag. We apply $C^{(r,s)}$ superrigidity to conclude that there exist a section ϕ of $P \to X$ with the same regularity as Φ, and $v \in V$ fixed by our parabolic subgroup, such that Φ is associated to (ϕ, v). We now restrict ourselves to some fiber F_x with a conull set of regular points. Since Φ is parallel, C^r regular, and defined on all of F_x, so is ϕ, i.e., there exist linearly independent parallel vector fields on F_x. Since $T \equiv 0$ (for the connection on F_x), parallel vector fields are commuting. The commuting vector fields now yield a locally free, transitive $\mathbb{R}^f$ action on F_x, allowing us to conclude F is homeomorphic to a torus.

4.1.2. Examples. It is possible to construct examples of bundles whose fibers F are any of those allowed by Theorem 4.1.

Example 4.1. Let $i\colon G \hookrightarrow H$ be the inclusion of G into H. It is possible to choose $H = SL(n)$ with n large enough so that $\mathbb{T}^f \subset H$ as a closed subgroup, and G and $\mathbb{T}^f$ commute. Let Γ be a cocompact torsion-free irreducible lattice in H. If $X = H/\Gamma$, then X admit a G action and a $\mathbb{T}^f$ action that commute. If $Y = X/\mathbb{T}^f$, we have a G action on Y that lifts to a G action on X. Ergodicity of G on X follows by irreducibility of Γ (Moore's Ergodicity Theorem), and the existence of a connection preserved by the G-action follows by reductiveness of X [11].

Example 4.2. The construction above can be generalized to show that F can be the homogeneous space of any compact Lie group K. Repeat the construction above, replacing $\mathbb{T}^f$ with any compact $K \subset H$ commuting with G. Thus, $X = H/\Gamma$ and $Y = X/K$. Now choose F such that K acts by diffeomorphisms on F, and form the associated fiber bundle E_F to X with fibers F, i.e.,

$$E_F = \frac{X \times F}{K}.$$

The G-action on E will be connection preserving, inheriting this property from the action of G on X. Also, if K acts transitively on F, then the G-action on E will be ergodic. Hence, by setting $F = K/K_0$, we obtain any homogeneous space of any compact Lie group. Of course, the fiber bundle which we obtain will generally be of a very large dimension, since H itself is quite large. It is an interesting problem to determine, for a given G, the possible types of fibers for a fiber bundle X of a given dimension.

We remark that this construction illustrates a fundamental difference between actions of a connected Lie group on a fiber bundle considered in Theorem 4.1 and actions of a lattice subgroup on a manifold considered in Theorem 4.4. In the example just constructed, we were able to exploit properties of compact principal K-bundles to obtain a fiber bundle whose fiber admit a transitive group of isometries. The analogous construction for the action of a lattice Γ on a manifold M would require a homomorphism $\Gamma \to \mathrm{Isom}(M)$, which by ergodicity, must have nonfinite image. Using the rigidity and arithmeticity theory of lattices in higher-rank semisimple Lie groups, such a homomorphism is possible only if the complexifications of G and $\mathrm{Isom}(M)$ have simple components which are isomorphic; see [8] and [12]. Taking $M = K$, where K is any compact Lie group, we see that this rarely possible.

4.2. Connection preserving actions of lattices. The following generalizes results in [6].

Theorem 4.4. *Let $\Gamma \subset G$ be an irreducible lattice in a higher-rank semisimple Lie group without compact factors. Suppose Γ acts ergodically on a compact n-dimensional manifold M preserving a volume and a smooth connection. Let $\alpha : \Gamma \times M \to SL(n)$ be the derivative cocycle with measurable algebraic hull L.*

(i) *If L is compact, then Γ acts isometrically on M preserving a smooth Riemannian metric.*
(ii) *If $L = SL(n)$ and $\pi(\Gamma) \subset SL(n)$ contains a lattice, where π is the superrigidity homomorphism, then M admits a torus as a finite affine cover.*

4.2.1. *Proof of Theorem* 4.4. In the first case, where L is compact, the proof follows as in Theorem 4.1. In the second case we employ Theorem 2.20 to deduce the existence of a C^r section ϕ of the flag bundle E_V associated to the principal $SL(n)$ bundle $P \to M$. We wish to show that modulo a finite subgroup, this decomposition comes from a smooth framing. Let $\{X_i\}$ be the measurable framing associated with this C^r decomposition. If x is a regular point, and C is a loop at x, then let P_C be parallel translation along the loop C. Thus

$$P_C(X_i(x)) = \sum_j H_{ij}(C)X_j(x), \tag{4}$$

for some matrix $H(C) = (H_{ij}(C))$. Since the measurable framing is associated to the C^r decomposition, we have $H_{ij}(C)$ is a diagonal matrix for all C, i.e.,

$$P_C(X_i(x)) = H_{ii}(C)X_i(x). \tag{5}$$

Lemma 4.5. *If $\gamma \in \Gamma$ such that x and γx are regular points, then*

$$\pi(\gamma)H(\gamma \circ C) = H(C)\pi(\gamma),$$

for any loop C at x, where π is the superrigidity homomorphism.

Proof. Using (5) and the fact that Γ preserves the connection, and hence commutes with parallel translation, we obtain

$$\begin{aligned}
\gamma P_C(X_i(x)) &= P_C(\gamma X_i(x)) \\
&= P_C\left(\epsilon(\gamma, x)\sum_k \pi_{ik}(\gamma)X_k(\gamma x)\right) \\
&= \sum_k \epsilon(\gamma, x)\pi_{ik}(\gamma)P_{\gamma\circ C}(X_k(\gamma x)) \\
&= \sum_{k,l} \epsilon(\gamma, x)\pi_{ik}(\gamma)H_{kl}(\gamma \circ C)X_l(\gamma x),
\end{aligned}$$

where $\epsilon(\gamma, x)$ is an element in the (compact) center of $SL(n)$, and $\pi: \Gamma \to SL(n)$ is the homomorphism from superrigidity, and $\pi_{ij}(\gamma)$ is the (i, j) entry of the matrix $\pi(\gamma)$.

On the other hand, we have

$$\begin{aligned}\gamma P_C(X_i(x)) &= \gamma \sum_k H_{ik}(C) X_k(x) \\ &= \sum_k H_{ik}(C)(\gamma X_k(x)) \\ &= \sum_k H_{ik}(C) \left(\epsilon(\gamma, x) \sum_l \pi_{kl}(\gamma) X_l(\gamma x) \right) \\ &= \epsilon(\gamma, x) \sum_{k,l} H_{ik}(C) \pi_{kl}(\gamma) X_l(\gamma x).\end{aligned}$$

By equating and summing these two equations, we obtain the desired conclusion.

Proposition 4.6. $H(C) = \pm I$.

Proof. Since the Γ action is ergodic and preserves a volume form, $H(C)$ must lie in $SL(n)$. Additionally, from (5) it follows that $H(C)$ is a diagonal matrix. Hence, the proposition follows once we demonstrate $H(C)$ is a scalar matrix.

Using Lemma 4.5 and diagonality of $H(C)$, we have

$$\pi_{ij}(\gamma) H_{jj}(\gamma \circ C) = H_{ii}(C) \pi_{ij}(\gamma). \tag{6}$$

for all $\gamma \in \Gamma$ such that x and γx are regular points. Since we can choose a finite set $\{\gamma_i\}$ of generators for Γ and an x such that $\{\gamma_i x\}$ is regular for all i, (6) holds for a fixed x and all γ. By Zariski density of $\pi(\Gamma)$, we have (6) holds for any matrix in $SL(n)$. By fixing j and varying i with any matrix with a nonzero (i, j) term, we see that all entries $H_{ii}(C)$ are equal. q.e.d.

So, modulo a finite subgroup, we have established the decomposition from a framing. Arguing as in [6], on some suitable covering, we actually have a framing, and as in Theorem 4.1, we conclude M must be a torus, thereby completing the proof of Theorem 4.4.

4.2.2. A few corollaries.

Corollary 4.7. *Let Γ be a lattice in $SL(n, \mathbb{R})$ acting ergodically on an n-dimensional manifold M preserving a volume density and a connection. Then the conclusion to Theorem* 4.4 *holds.*

Proof. If L is not compact, then from superrigidity, we know that π must be a surjection with finite kernel, hence $\pi(\Gamma)$ is a lattice. The result now follows from Theorem 4.4.

Corollary 4.8. *Suppose $SL(n, \mathbb{Z})$, $n \geq 3$, acts ergodically on an n-dimensional manifold M preserving a connection and a volume density. Then M admits a torus as a finite affine cover.*

Proof. Since $SL(n, \mathbb{Z})$ is a noncocompact lattice in $SL(n, \mathbb{R})$, the first case in Theorem 4.4 is not possible. That the second case holds follows from Corollary 4.7.

Bibliography

[1] A. Borel & N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton Univ. Press, Princeton, NJ, 1980.

[2] A. Capozzoli, *Bundle automorphisms for fiber G-structures of finite type*, Geometriae Dedicata, to appear.

[3] D. G. Ebin, *The manifold of riemannian metrics*, Global Analysis Proc. Sympos. Pure Math., Vol. 15, Amer. Math. Soc., Providence, RI, 1970, 11–40.

[4] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[5] R. Feres, *Actions of discrete linear groups and Zimmer's conjecture*, preprint.

[6] ———, *Connections preserving actions of lattices in* $SL(n, \mathbb{R})$, Israel J. Math. **79** (1992) 1–21.

[7] S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, Ann. of Math. (2) **135** (1992) 361–410.

[8] F.E.A. Johnson, *On the existence of irreducible discrete subgroups in isotypic Lie groups of classical type*, Proc. London Math. Soc. **56** (1988) 51–77.

[9] A. Katok & J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume preserving actions*, preprint.

[10] ———, *Local rigidity for certain groups of toral automorphisms*, Israel J. Math. **75** (1991) 203–241.

[11] S. Kobayashi & Katsumi Nomizu, *Foundations of differential geometry*, Wiley, New York, 1963.

[12] G. A. Margulis, *Discrete subgroups of lie groups*, Springer, New York, 1991.

[13] Peter Walters, *An introduction to ergodic theory*, Springer, New York, 1982.

[14] Robert J. Zimmer, *Topological superrigidity*, preprint.

[15] ———, *Ergodic theory and semisimple groups*, Birkhauser, Boston, 1984.

[16] ———, *Ergodic theory and the automorphism group of a G-structure*, Group Representations, Ergodic Theory, Operator Algebras, and Math. Physics (C. C. Moore, ed.), Springer, New York, 1987, 247–278.

[17] ———, *Actions of simple Lie groups*, Workshop on Lie Groups, Ergodic Theory and Geometry and Problems in Geometric Rigidity (Berkeley, CA), Math. Sci. Res. Inst., Berkeley, CA, 1992.

UNIVERSITY OF MICHIGAN